Reflection Positivity of $\mathcal{N} = 1$ Wess-Zumino model on the lattice with exact U(1)$_R$ symmetry

Yoshio Kikukawa[*] and Kouta Usui[†]
(Dated: January 17, 2012)

By using overlap Majorana fermions, the $\mathcal{N} = 1$ chiral multiple can be formulated so that the supersymmetry is manifest and the vacuum energy is cancelled in the free limit, thanks to the bilinear nature of the free action. It is pointed out, however, that in this formulation the reflection positivity seems to be violated in the bosonic part of the action, although it is satisfied in the fermionic part. It is found that the positivity of the spectral density of the bosonic two-point correlation function is ensured only for the spacial momenta $a|p_k| \lesssim 1.84$ $(k = 1, 2, 3)$. It is then argued that in formulating $\mathcal{N} = 1$ Wess-Zumino model with the overlap Majorana fermion, one may adopt a simpler nearest-neighbor bosonic action, discarding the free limit manifest supersymmetry. The model still preserves the would-be U(1)$_R$ symmetry and satisfies the reflection positivity.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

The chiral multiplet of $\mathcal{N} = 1$ supersymmetry[1] can be formulated on the lattice so that the supersymmetry is preserved and the vacuum energy is cancelled in the free limit, thanks to the bilinear nature of the free action. By using overlap (Majorana) fermion [2–4] for the fermionic component, species doublers[5–7] are successfully removed and U(1)$_R$ symmetry can be maintained at the same time[8–10]. With this chiral multiplet, one may formulate lattice $\mathcal{N} = 1$ Wess-Zumino model with exact U(1)$_R$ symmetry[11–16]. A numerical study of this lattice $\mathcal{N} = 1$ Wess-Zumino model has recently been reported in [17].

The purpose of this short article is, however, to show that in this formulation of the chiral multiplet, the reflection positivity[18–22] seems to be violated in the bosonic part of the action, although it is satisfied in the fermionic part, as shown recently in [23]. We will also examine the spectral density of the bosonic two-point correlation function (cf. [24]). It is found that the positivity of the spectral density is ensured only for the momenta $a|p_k| \lesssim 1.84$ $(k = 1, 2, 3)$, and the mode with a negative density appears at the energy as low as $aE \simeq 0.69$ for the momenta $a\boldsymbol{p} = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi)$.

We will then argue that in formulating the lattice $\mathcal{N} = 1$ Wess-Zumino model with the overlap (Majorana) fermion, one may adopt the simpler nearest-neighbor bosonic action, discarding the free limit manifest supersymmetry. The model so constructed still preserves the U(1)$_R$ symmetry and satisfies the reflection positivity.

This paper is organized as follows. In section II, we review briefly the $\mathcal{N} = 1$ chiral multiple on the lattice formulated with overlap Majorana fermion. In section III, we show that

the standard way to prove the reflection positivity does not work in the bosonic part of the action. The spectral density of the bosonic two-point correlation function is also examined. In section IV, we show that it is possible to formulate lattice Wess-Zumino model which possesses both the reflection positivity and the exact U(1)$_R$ symmetry, by adopting the simpler nearest-neighbor bosonic action. Section V is devoted to discussion.

II. $\mathcal{N} = 1$ CHIRAL MULTIPLE WITH OVERLAP MAJORANA FERMION

The action of the free $\mathcal{N} = 1$ chiral multiplet is given by

$$S_0 = a^4 \sum_x \Big\{ \frac{1}{2} \chi^T C D_1 \chi + \phi^* D_1^2 \phi + F^* F + \frac{1}{2} \chi^T C D_2 \chi + F D_2 \phi + F^* D_2 \phi^* \Big\}. \quad (1)$$

In this expression, we have used a decomposition of the overlap Dirac operator [2][3], $D = D_1 + D_2$, where

$$D_1 = \frac{1}{2} \gamma_\mu (\partial_\mu^* + \partial_\mu)(A^\dagger A)^{-1/2}, \quad (2)$$

$$D_2 = \frac{1}{a} \Big\{ 1 - (1 + \frac{1}{2} a^2 \partial_\mu^* \partial_\mu)(A^\dagger A)^{-1/2} \Big\}, \quad (3)$$

and

$$A = 1 - a D_{\rm w}, \quad D_{\rm w} = \frac{1}{2} \Big\{ \gamma_\mu (\partial_\mu^* + \partial_\mu) - a \partial_\mu^* \partial_\mu \Big\}. \quad (4)$$

Note that D_1 and D_2 have different spin structures with respect to spinor space. In particular, we have $\{\gamma_5, D_1\} = 0$ and $[\gamma_5, D_2] = 0$. In terms of this decomposition, the Ginsparg-Wilson relation $\gamma_5 D + D \gamma_5 = a D \gamma_5 D$ [8] is expressed as

$$2 D_2 = a(-D_1^2 + D_2^2), \quad (5)$$

and as a consequence, we have relations

$$\gamma_5 (1 - \frac{1}{2} a D) \gamma_5 (1 - \frac{1}{2} a D) = 1 - \frac{1}{2} a D_2, \quad (6)$$

$$\gamma_5 (1 - \frac{1}{2} a D) \gamma_5 D = D_1. \quad (7)$$

[*]Institute of Physics, University of Tokyo, Tokyo, 153-8902, Japan; Electronic address: kikukawa@hep1.c.u-tokyo.ac.jp
[†]Department of physics, University of Tokyo, 113-0033, Japan
Institute for the Physics and Mathematics of the Universe (IPMU), the University of Tokyo, Chiba 277-8568, Japan; Electronic address: kouta@hep-th.phys.s.u-tokyo.ac.jp

It is also understood that the 4×4 identity matrix in operators D_1^2 and D_2 is omitted when these operators are acting on bosonic fields.

It is straightforward to see that the above free action S_0 is invariant under "lattice $\mathcal{N} = 1$ supersymmetry":

$$\begin{aligned}
\delta_\epsilon \chi &= -\sqrt{2} P_+ (D_1 \phi + F)\epsilon - \sqrt{2} P_- (D_1 \phi^* + F^*)\epsilon, \\
\delta_\epsilon \phi &= \sqrt{2}\epsilon^T C P_+ \chi, \quad \delta_\epsilon \phi^* = \sqrt{2}\epsilon^T C P_- \chi, \\
\delta_\epsilon F &= \sqrt{2}\epsilon^T C D_1 P_+ \chi, \quad \delta_\epsilon F^* = \sqrt{2}\epsilon^T C D_1 P_- \chi,
\end{aligned} \quad (8)$$

where ϵ is a 4 component Grassmann parameter. We also note that the free action S_0 possesses three types of U(1) symmetry [10]. The first is a rather trivial one acting only on bosonic fields and is defined by the transformation:

$$\begin{aligned}
\delta_\alpha \chi &= 0, \\
\delta_\alpha \phi &= i\alpha\phi, \\
\delta_\alpha F &= -i\alpha F,
\end{aligned} \quad (9)$$

where α is an infinitesimal real parameter. The second one is nothing but the chiral symmetry introduced by Lüscher,

$$\delta_\alpha \chi = i\alpha\gamma_5 (1 - \frac{1}{2}aD)\chi, \quad (10)$$

Thirdly, somewhat surprisingly, the *bosonic* sector of S_0 possesses a U(1) symmetry analogous to eq. (10):

$$\begin{aligned}
\delta_\alpha \phi &= +i\alpha\{(1 - \frac{1}{2}aD_2)\phi - \frac{1}{2}aF^*\}, \\
\delta_\alpha F &= +i\alpha\{(1 - \frac{1}{2}aD_2)F - \frac{1}{2}aD_1^2\phi^*\}
\end{aligned} \quad (11)$$

due to the Ginsparg-Wilson relation. The lattice action S_0 is not invariant under a uniform rotation of the complex phase of bosonic fields, ϕ, F, due to the presence of terms $FD_2\phi$ and $F^* D_2 \phi^*$. The above provides a lattice counterpart of this uniform phase rotation of bosonic fields under which the free action S_0 is invariant. Using a linear combination of the above three U(1) symmetries, it is possible to define the U(1)$_R$ symmetry [10] in the interacting system.

$$\begin{aligned}
\delta_\alpha \chi &= +i\alpha\gamma_5(1 - \frac{1}{2}aD)\chi, \\
\delta_\alpha \phi &= -3i\alpha\phi + i\alpha\{(1 - \frac{1}{2}aD_2)\phi - \frac{1}{2}aF^*\}, \\
\delta_\alpha F &= +3i\alpha F + i\alpha\{(1 - \frac{1}{2}aD_2)F - \frac{1}{2}aD_1^2\phi^*\}.
\end{aligned} \quad (12)$$

III. VIOLATION OF THE REFLECTION POSITIVITY IN THE BOSONIC PART

A. Reflection positivity condition

In this subsection, we will formulate the reflection positivity condition. It has been rigorously shown that the lattice theory satisfying the reflection positivity condition corresponds to the quantum theory with unitary time evolution [18–20].

Here we consider the generic case in which there are both a bosonic field ϕ and a fermionic field ψ. Let us assume that $S(\phi, \psi, \bar{\psi})$ is the action of a lattice model[36] and its partition function Z is given by the path integration

$$Z = \int [\mathcal{D}\phi \mathcal{D}\phi^*][\mathcal{D}\psi \mathcal{D}\bar{\psi}]\, \mathrm{e}^{-S(\phi,\psi,\bar{\psi})}. \quad (13)$$

We set the lattice spacing a to be unity, and assume the finite volume hypercubic lattice $\Lambda = \{-L+1, -L+2, \ldots, L-1, L\}^d \subset \mathbb{Z}^d$. We impose the *anti*-periodic boundary condition in the time direction for the fermionic field ψ, while the periodic boundary condition for the bosonic field ϕ. In the spacial directions, periodic boundary conditions are imposed for both fields.

To formulate the reflection positivity condition, we first introduce the time reflection operator θ as follows. For each site $x = (t, \boldsymbol{x}) \in \Lambda$, we denote $\theta x = (-t+1, \boldsymbol{x})$. This is the time reflection with respect to the $t = 1/2$ plane. We define the operation of θ for bosonic fields as

$$(\theta\phi)(x) = \phi(\theta x) \quad (14)$$

and for functions of bosonic fields $\mathcal{F}(\phi)$ as

$$(\theta\mathcal{F})(\phi) = \mathcal{F}^*(\theta\phi), \quad (15)$$

where * means complex conjugation. For fermionic fields, the θ reflection is defined as

$$(\theta\bar{\psi})(x) = \gamma_0 \psi(\theta x), \quad (16)$$
$$(\theta\psi)(x) = \bar{\psi}(\theta x)\gamma_0. \quad (17)$$

We extend this θ operation to the whole field algebra $\mathcal{A}$. We define the field algebra $\mathcal{A}$, the algebra of observables, as the Grassmann algebra generated by the fermionic fields with the coefficients of the continuous functions of bosonic fields which are integrable with respect to the bosonic Gaussian functional measure. For $\mathcal{F}, \mathcal{G} \in \mathcal{A}$, the θ operation is defined by the relations

$$\theta(\mathcal{F}\mathcal{G}) = \theta(\mathcal{G})\theta(\mathcal{F}), \quad (18)$$
$$\theta(\alpha\mathcal{F} + \beta\mathcal{G}) = \alpha^*\theta(\mathcal{F}) + \beta^*\theta(\mathcal{G}). \quad (19)$$

For instance, if $\mathcal{F}$ has the form of

$$\begin{aligned}
\mathcal{F}(\phi, \psi, \bar{\psi}) =& f(\phi)\bar{\psi}_{a_1}(x_1)\ldots\bar{\psi}_{a_n}(x_n)\times \\
& \times \psi_{b_1}(y_1)\ldots\psi_{b_m}(y_m),
\end{aligned} \quad (20)$$

its θ reflection should be

$$\begin{aligned}
\theta(\mathcal{F})(\phi, \psi, \bar{\psi}) =& f^*(\theta\phi)(\bar{\psi}\gamma_0)_{b_m}(\theta y_m)\ldots(\bar{\psi}\gamma_0)_{b_1}(\theta y_1)\times \\
& \times (\gamma_0\psi)_{a_n}(\theta x_n)\ldots(\gamma_0\psi)_{a_1}(\theta x_1).
\end{aligned} \quad (21)$$

Let Λ_+ (resp. Λ_-) be the set of lattice sites with positive (resp. non-positive) time components, and $\mathcal{A}_\pm$ be the subalgebras of $\mathcal{A}$, which depends only upon fields on $\Lambda_\pm$. In this notation, θ is a map from $\Lambda_\pm$ into $\Lambda_\mp$ and from $\mathcal{A}_\pm$ into $\mathcal{A}_\mp$.

Reflection positivity condition is defined through this θ map. For a lattice theory with the expectation functional $\langle \cdot \rangle$ defined for $\mathcal{F} \in \mathcal{A}$ as

$$\langle \mathcal{F} \rangle = \frac{1}{Z} \int [\mathcal{D}\phi\mathcal{D}\phi^*][\mathcal{D}\psi\mathcal{D}\bar{\psi}] \, \mathrm{e}^{-S(\phi,\psi,\bar{\psi})} \mathcal{F}(\phi,\psi,\bar{\psi}), \quad (22)$$

we say the theory is reflection positive with respect to θ if any function $\mathcal{F}_+ \in \mathcal{A}_+$ fulfills the inequality

$$\langle \theta(\mathcal{F}_+)\mathcal{F}_+ \rangle \geq 0. \quad (23)$$

B. Reflection positivity of the free overlap boson

In this subsection, we investigate the reflection positivity of the bosonic sector of the free chiral multiplet (1). It will be shown in the following that the bosonic sector does not seem to satisfy the reflection positivity condition. After integrating out the auxiliary field F, we have the overlap boson system which is defined through the lattice action on Λ

$$S_b(\phi) = \sum_{x \in \Lambda} \phi^*(x)\square_\Lambda\phi(x), \quad (24)$$

where we have defined

$$\square_\Lambda(x,y) = \sum_{n \in \mathbb{Z}^4} \square(x+2nL,y), \quad (25)$$

and

$$D^\dagger D = \left\{ 1 - (1 + \tfrac{1}{2}\partial_\mu^*\partial_\mu)(A^\dagger A)^{-1/2} \right\} \quad (26)$$

$$= \square \cdot \hat{1} \quad (27)$$

with $\hat{1}$ being the unit spinor matrix. The operator $\square$ given above is the bosonic overlap operator on $\mathbb{Z}^d$ and $\square_\Lambda$ is that on Λ with periodic boundary conditions. The field algebra $\mathcal{A}$ of this overlap boson system is defined as the set of all continuous functions of bosonic field configulations $\phi = \{\phi(x)\}_{x \in \Lambda}$, which are integrable with respect to the bosonic Gaussian measure

$$[\mathcal{D}\phi][\mathcal{D}\phi^*] \, \mathrm{e}^{-S_b(\phi)}. \quad (28)$$

The expectation of this theory is defined by the bosonic path integration

$$\langle \mathcal{F} \rangle = \frac{1}{Z} \int [\mathcal{D}\phi][\mathcal{D}\phi^*] \, \mathrm{e}^{-S_b(\phi)} \mathcal{F}(\phi), \qquad \mathcal{F} \in \mathcal{A}. \quad (29)$$

The standard way of investigating the reflection positivity of lattice field theory is to prove that the action can be written in the form of

$$-S_b(\phi) = B(\phi) + \theta(B)(\phi) + \sum_s \theta(C_s)(\phi)C_s(\phi), \quad (30)$$

with $B, C_s \in \mathcal{A}_+$, where in the third term C_s are elements of $\mathcal{A}_+$ parametrized by some discrete parameter s [20]. To see

that the equation (30) indeed implies the reflection positivity (23), we first note that for an arbitrary $\mathcal{F}_+ \in \mathcal{A}_+$,

$$\langle \theta(\mathcal{F}_+)\mathcal{F}_+ \rangle_0$$

$$= \int [\mathcal{D}\phi][\mathcal{D}\phi^*]\theta(\mathcal{F}_+)\mathcal{F}_+$$

$$= \int \prod_{x \in \Lambda_+} d\phi(x)d\phi^*(x) \, \mathcal{F}_+(\phi) \int \prod_{x \in \Lambda_-} d\phi(x)d\phi^*(x) \, \theta(\mathcal{F}_+)(\phi)$$

$$= \left| \int \prod_{x \in \Lambda_+} d\phi(x)d\phi^*(x) \, \mathcal{F}_+(\phi) \right|^2 \geq 0. \quad (31)$$

If the action is given in the form of (30), we obtain for all $\mathcal{F} \in \mathcal{A}_+$,

$$\left\langle \mathrm{e}^{-S_b}\theta(\mathcal{F}_+)\mathcal{F}_+ \right\rangle_0$$

$$= \left\langle \mathrm{e}^{B+\theta(B)+\sum_s \theta(C_s)C_s}\theta(\mathcal{F}_+)\mathcal{F}_+ \right\rangle_0$$

$$= \left\langle \theta(\mathrm{e}^B) \, \mathrm{e}^B \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_s \theta(C_s)C_s \right)^n \theta(\mathcal{F}_+)\mathcal{F}_+ \right\rangle_0$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{s_1 \dots s_n} \left\langle \theta(\mathrm{e}^B) \, \mathrm{e}^B \, \theta(C_{s_1})C_{s_1} \dots \theta(C_{s_n})C_{s_n} \, \theta(\mathcal{F}_+)\mathcal{F}_+ \right\rangle_0$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{s_1 \dots s_n} \left\langle \theta\left(\mathrm{e}^B C_{s_1} \dots C_{s_n} \mathcal{F}_+ \right) \mathrm{e}^B C_{s_1} \dots C_{s_n} \mathcal{F}_+ \right\rangle_0. \quad (32)$$

This last expression is clearly positive from (31). This immediately implies the reflection positivity because

$$\langle \theta(\mathcal{F}_+)\mathcal{F}_+ \rangle = \frac{\left\langle \mathrm{e}^{-S_b}\theta(\mathcal{F}_+)\mathcal{F}_+ \right\rangle_0}{\left\langle \mathrm{e}^{-S_b} \right\rangle_0} \geq 0, \qquad \mathcal{F}_+ \in \mathcal{A}_+. \quad (33)$$

We note that the third term in (30) may be given by an integration over a continuous parameter s as

$$\int ds \, \theta(C_s)(\phi)C_s(\phi), \qquad C_s \in \mathcal{A}_+. \quad (34)$$

This type of the action appears in the case of overlap fermions. See Ref.[23] for detail.

Therefore, to prove the reflection positivity of the 'overlap boson' system reduces to find the decomposition of the action (24) into (30). We first note that S_b can be written as

$$S_b = \sum_{x,y \in \Lambda_+} \phi^*(x)\square(x,y)\phi(y) + \sum_{x,y \in \Lambda_-} \phi^*(x)\square(x,y)\phi(y)$$

$$+ 2 \sum_{x \in \Lambda_+, y \in \Lambda_-} \phi^*(x)\square(x,y)\phi(y), \quad (35)$$

where $\square(x,y)$ is the kernel of the operator $\square$ on Λ. To establish the decomposition (30), we should find that (i) the second term is the θ reflection of the first term, and that (ii) the last term is written in the form of

$$-\int \theta(C_s)C_s \, ds \quad (36)$$

for some $C_s \in \mathcal{A}_+$ parametrized by some parameter s. Note that this second condition is equivalent to say that

$$-\sum_{x \in \Lambda_+, y \in \Lambda_-} \phi^*(x)\square(x,y)\phi(y) = \int f(s)\,\theta(C_s)(\phi)\,C_s(\phi)\,ds \tag{37}$$

for some *non-negative* function $f(s)$. In this bosonic system, while (i) holds true, the property (ii) breaks down, as will be shown below.

To show this, we will derive the spectral representation of the kernel $\square(x,y)$. First, the Fourier transformation $\square(p)$ is given by:

$$\square(p) = 1 - \frac{1 - \sum_\mu(1 - \cos p_\mu)}{\sqrt{\sum_\mu \sin^2 p_\mu + \left[1 - \sum_\mu(1 - \cos p_\mu)\right]^2}}$$

$$= 1 + \frac{b(\boldsymbol{p}) - \cos p_0}{\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos p_0}}, \tag{38}$$

where

$$a(\boldsymbol{p}) = 1 + \sum_j \sin^2 p_j + b(\boldsymbol{p})^2, \tag{39}$$

$$b(\boldsymbol{p}) = \sum_j(1 - \cos p_j). \tag{40}$$

From this formula we obtain the following three-space representation of the kernel $\square(x,y)$,

$$\square(x,y)\Big|_{x_0 \neq y_0} = \int_{[-\pi,\pi]^3} \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} I(x_0 - y_0; \boldsymbol{p}), \tag{41}$$

where we have defined

$$I(x_0 - y_0; \boldsymbol{p}) = \int_{-\pi}^{\pi} \frac{dp_0}{2\pi}\,e^{ip_0(x_0-y_0)} \frac{b(\boldsymbol{p}) - \cos p_0}{\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos p_0}}. \tag{42}$$

We can transform $I(x_0; \boldsymbol{p})$ into a spectral representation by applying Cauchy's integration theorem to the contour in the complex p_0 plane drawn in the FIG.1. In the FIG.1, the
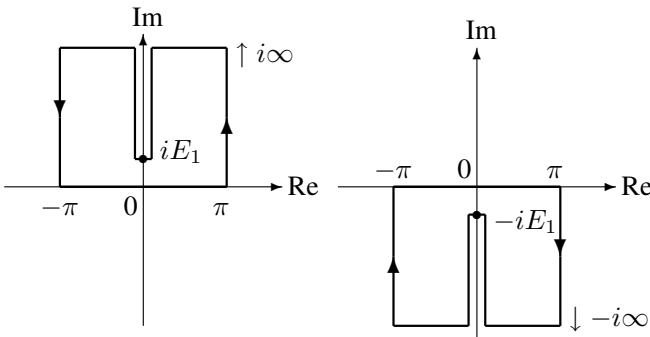

FIG. 1: Complex integration contours

original integration contour for p_0 is the interval $[-\pi, \pi]$ on the real axis, and E_1 is defined as the positive solution of

$$2b(\boldsymbol{p})\cosh E_1 - a(\boldsymbol{p}) = 0. \tag{43}$$

In the case of $x_0 > 0$, we use the left contour and, in the case of $x_0 < 0$, we use the right. Note that the contributions coming from integrations along the edge with infinite (or minus infinite) imaginary part (upper edge in the case of $x_0 > 0$, lower in the case of $x_0 < 0$) vanishes. Furthermore, because of the periodicity of the integrand, the contributions coming from integration along the edge whose real part is π (the right edge of the contour) and from integration along the edge $-\pi$ (left edge of the contour) cancel each other. In this way, the original p_0 integration on the interval $[-\pi, \pi]$ can be expressed as an integration on the interval $[iE_1, i\infty]$ (or $[-iE_1, -i\infty]$) along the imaginary axis:

$$I(x_0; \boldsymbol{p})\Big|_{x_0 \neq 0} = \int_{-\pi}^{\pi} \frac{dp_0}{2\pi}\,e^{ip_0 x_0} \frac{b(\boldsymbol{p}) - \cos p_0}{\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos p_0}}$$

$$= \int_{E_1}^{\infty} \frac{dE}{\pi}\,e^{-E|x_0|} \frac{b(\boldsymbol{p}) - \cosh E}{\sqrt{2b(\boldsymbol{p})\cosh E - a(\boldsymbol{p})}}. \tag{44}$$

Since the kernel of the operator on the finite lattice $\square_\Lambda(x,y)$ is defined as (25), it is straightforward to derive the following spectral representation of $\square_\Lambda$,

$$\square_\Lambda(x,y)\Big|_{x_0 > y_0}$$

$$= \frac{1}{(2L)^3} \sum_{\boldsymbol{p}} e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \sum_{n_0 \in \mathbb{Z}} I(x_0 + 2n_0 L - y_0; \boldsymbol{p})$$

$$= \frac{1}{(2L)^3} \sum_{\boldsymbol{p}} e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \int_{E_1}^{\infty} \frac{dE}{\pi} \frac{1}{1 - e^{-2EL}} \times$$

$$\times \left[e^{-E(x_0-y_0)} + e^{-E(2L-x_0+y_0)}\right] \frac{b(\boldsymbol{p}) - \cos p_0}{\sqrt{a(\boldsymbol{p}) - 2b(\boldsymbol{p})\cos p_0}}. \tag{45}$$

The second term represents a finite volume effect.

Now we can see the property (37) does not hold. From eq.(45), we obtain

$$-\sum_{x \in \Lambda_+, y \in \Lambda_-} \phi^*(x)\square_\Lambda(x,y)\phi(y)$$

$$= \frac{1}{(2L)^3} \sum_{\boldsymbol{p}} \int_{E_1}^{\infty} \frac{dE}{\pi} \frac{1}{1 - e^{-2EL}} \frac{\cosh E - b(\boldsymbol{p})}{\sqrt{2b(\boldsymbol{p})\cosh E - a(\boldsymbol{p})}} \times$$

$$\times \sum_{x \in \Lambda_+, y \in \Lambda_-} \left\{e^{-E(x_0-y_0)}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\phi^*(x)\phi(y) +\right.$$

$$\left. + e^{-2EL}e^{+E(x_0-y_0)}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\phi^*(x)\phi(y)\right\}$$

$$= \frac{1}{(2L)^3} \sum_{\boldsymbol{p}} \int_{E_1}^{\infty} \frac{dE}{\pi} \frac{1}{1 - e^{-2EL}} \frac{\cosh E - b(\boldsymbol{p})}{\sqrt{2b(\boldsymbol{p})\cosh E - a(\boldsymbol{p})}} \times$$

$$\times \left\{\theta(C_{E,\boldsymbol{p}})(\phi)(C_{E,\boldsymbol{p}})(\phi) +\right.$$

$$\left. + \theta(e^{-EL}C_{-E,\boldsymbol{p}})(\phi)(e^{-EL}C_{-E,\boldsymbol{p}})(\phi)\right\}, \tag{46}$$

where we define

$$C_{E,\boldsymbol{p}}(\phi) = \sum_{x \in \Lambda_+} e^{-Ex_0} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi^*(x) \in \mathcal{A}_+. \qquad (47)$$

In this case, $(E, \boldsymbol{p})$ plays a role of the parameter s in (37). For the condition (37) to be satisfied, the coefficient factor $\cosh E - b(\boldsymbol{p})$ should be non-negative for any $(E, \boldsymbol{p})$ satisfying $E_1 \leq E$, but this is *not* the case. In fact, $\cosh E - b(\boldsymbol{p})$ can become *both* positive and negative in general, which prevents us from proving the reflection positivity.

C. Källén-Lehmann representation of the free overlap boson propagator

In the previous section, we have shown that the standard way of proving the reflection positivity does not work for the overlap boson. In this section, we will investigate the Källén-Lehmann representation of the propagator. Here, we will observe that the spectral density function, which is expected to be positive (non-negative) for unitary quantum theories, is not a positive function. This implies that the overlap boson system has pathological spectrum of energy momentum operators.

The spectral density $\rho(E, \boldsymbol{p})$ is geven in the the Euclidean version of Källén-Lehmann representation by

$$\Delta_+(x, y) := \langle \phi(x)^* \phi(y) \rangle \Big|_{x_0 > y_0}$$
$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \int_0^\infty \frac{dE}{\pi} e^{-E(x_0 - y_0)} e^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \rho(E, \boldsymbol{p}). \tag{48}$$

In the present case, one can explicitly estimate the propagator and the spectral density $\rho(E, \boldsymbol{p})$ by using the Fourier transformation,

$$\Delta_+(x, y) = \frac{1}{\Box}(x, y) \Big|_{x_0 > y_0}$$
$$= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{\Box(p)}. \tag{49}$$

The result is

$$\rho(E, \boldsymbol{p}) = (\text{proportional to delta function}) +$$
$$+ \frac{(\cosh E - b(\boldsymbol{p}))\sqrt{2b(p)\cosh E - a(\boldsymbol{p})}}{\cosh^2 E - a(\boldsymbol{p}) + b(\boldsymbol{p})^2} \theta(E - E_1). \tag{50}$$

This formula is derived in Appendix A. The second term, continuous spectrum, is not positive (non-negative) because of the factor $\cosh E - b(\boldsymbol{p})$, which is exactly the same one as appeared in (46).

The fact that the positivity of the spectral density breaks down is an indirect but strong circumstantial evidence that the overlap boson system does not define a quantum mechanical system with physically satisfactory energy-momentum spectrum. Further, it is very probable that the overlap boson system breaks the reflection positivity condition.

In fact, one can prove in mathematically rigorous manner [25] that a lattice theory satisfying the reflection positivity condition, in addition to some technical assumptions which are satisfied by the overlap boson system, must have the non-negative spectral density. The proof goes as follows. For a detail, see [25]. Suppose that the reflection positivity is satisfied, and take the infinite volume limit $\Lambda \to \mathbb{Z}^d$. Then, from the reflection positivity condition, the Hilbert space of state vectors is constructed and for each $\mu = 0, 1, \ldots, d-1$, momentum operator P_μ acting on this Hilbert space is defined as an infinitesimal self adjoint generator of translation in each direction. We denote $P_0 = H$ and $\boldsymbol{P} = (P_1, \ldots, P_{d-1})$. Since translations in different directions commute, P_μ and P_ν commute with each other if $\mu \neq \nu$. Therefore $(H, \boldsymbol{p})$ possesses complete orthonormal set of simultaneous eigenvectors $|E, \boldsymbol{p}\rangle$ with $E \geq 0$, $\boldsymbol{p} \in [-\pi, \pi)^{d-1}$. Let $|0\rangle$ be $|0, \boldsymbol{0}\rangle$. Through momentum operators $(H, \boldsymbol{P})$, field operator at $x = (x_0, \boldsymbol{x}) \in \mathbb{Z}^d$, $\hat{\phi}(x)$ is related to the field operator at the origin $\hat{\phi}(0)$ by the relation

$$\hat{\phi}(x) = e^{-x_0 H} e^{-i\boldsymbol{P}\cdot\boldsymbol{x}} \hat{\phi}(0) e^{x_0 H} e^{i\boldsymbol{P}\cdot\boldsymbol{x}}. \tag{51}$$

The two point function is expressed in terms of these ingredients of Hilbert space,

$$\langle \phi(x)^* \phi(y) \rangle = \langle 0| \hat{\phi}(x)^\dagger \hat{\phi}(y) |0\rangle. \tag{52}$$

By the translational invariance and (51), and by inserting the identity in the form of

$$1 = \int_{[0,\infty) \times [-\pi,\pi)^{d-1}} dE \, d^3\boldsymbol{p} \, |E, \boldsymbol{p}\rangle\langle E, \boldsymbol{p}|, \tag{53}$$

we obtain

$$\Delta_+(x, y) = \langle 0|\hat{\phi}(x)^\dagger \hat{\phi}(y)|0\rangle \Big|_{x_0 > y_0}$$
$$= \langle 0|\hat{\phi}(x-y)^\dagger \hat{\phi}(0)|0\rangle$$
$$= \langle 0|\hat{\phi}(0)^\dagger e^{-(x_0-y_0)H} e^{-i\boldsymbol{P}\cdot(\boldsymbol{x}-\boldsymbol{y})} \hat{\phi}(0)|0\rangle$$
$$= \int_{[0,\infty) \times [-\pi,\pi)^{d-1}} d|\langle 0|\hat{\phi}(0)|E, \boldsymbol{p}\rangle|^2$$
$$e^{-E(x_0-y_0)} e^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}, \tag{54}$$

where the integration measure is formally given by

$$d|\langle 0|\hat{\phi}(0)|E, \boldsymbol{p}\rangle|^2 = d^3\boldsymbol{p} \, dE \, |\langle 0|\hat{\phi}(0)|E, \boldsymbol{p}\rangle|^2. \tag{55}$$

Even though the above computation is rather formal, the measure (55) has mathematically rigorous meaning as a d-dimensional Borel measure supported on $[0, \infty) \times [-\pi, \pi)^{d-1}$. In fact, one can derive (54) in mathematically rigorous manner. It is well known in measure theory that any measure μ on $\mathbb{R}^d$ can be uniquely decomposed into two parts

$$\mu = \mu_s + \mu_{\text{abs}} \tag{56}$$

where μ_s is singular and μ_{abs} is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ (Lebesgue decomposition theorem). In more ordinary expression in physics, this

theorem states that if we write

$$d|\langle 0|\hat{\phi}(0)|E, \boldsymbol{p}\rangle|^2 = \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{dE}{\pi} \rho(E, \boldsymbol{p}), \qquad (57)$$

$\rho(E, \boldsymbol{p})$ can be written corresponding to (56) as

$$\rho(E, \boldsymbol{p}) = (\text{singular part}) + \rho_c(E, \boldsymbol{p}), \qquad (58)$$

where $\rho_c(E, \boldsymbol{p})$ is *non-negative* integrable function. This is not the case for the overlap boson.

D.   Estimation of the violation in the momentum space

From the explicit form of the spectral density (50), we can find where in the Brillouin zone the reflection positivity is violated. One notes that there is the region in the spacial Brillouin zone where the spectral density $\rho(E, \boldsymbol{p})$ can not become negative. Let us call this region $\mathcal{S}$. The region $\mathcal{S}$ is characterized by the condition that the negative value of $\cosh E - b(\boldsymbol{p})$ be avoided. The necessary and sufficient condition on spacial momenta $\boldsymbol{p}$ to avoid negative $\cosh E - b(\boldsymbol{p})$ is that

$$\cosh E - b(\boldsymbol{p}) \geq 0, \qquad \forall E \geq E_1, \qquad (59)$$

which is equivalent to the condition $\cosh E_1 \geq b(\boldsymbol{p})$

$$\text{i.e.} \quad \frac{1 + \sum_k \sin^2 p_k - (\sum_k (1 - \cos p_k))^2}{2\sum_k (1 - \cos p_k)} \geq 0, \qquad (60)$$

or, equivalently,

$$1 + \sum_k \sin^2 p_k - b(\boldsymbol{p})^2 \geq 0. \qquad (61)$$

Then, define

$$\mathcal{S} = \left\{ \boldsymbol{p} \in [-\pi, \pi)^{d-1} \ : \ 1 + \sum_k \sin^2 p_k - b(\boldsymbol{p})^2 \geq 0 \right\}, \qquad (62)$$

where d is the spacetime dimension.

Now let us estimate the size of $\mathcal{S}$ to investigate whether we can ignore the violation of the reflection positivity or not. In the case of $d = 4$, there are three spacial momentum components. First, we consider the case in which $p_1 = p_2 = p_3 =: p$. In this direction, the safe momentum region has the extent

$$-1.84 \lesssim p \lesssim 1.84. \qquad (63)$$

Second, we consider another direction $p_1 = p$, $p_2 = p_3 = 0$. In this case, in the safe region $\mathcal{S}$, p is restricted by

$$-2.23 \lesssim p \lesssim 2.23. \qquad (64)$$

These regions are a little bit lager than $[-\pi/2, \pi/2]^{d-1}$.

When the spacial momenta $\boldsymbol{p}$ does not belong to $\mathcal{S}$, the spectral density $\rho(E, \boldsymbol{p})$ has to become negative on the energy interval $E_1 \leq E < E_c$, where E_1 and E_c are determined by

$$\cosh E_1 = \frac{a(\boldsymbol{p})}{2b(\boldsymbol{p})}, \qquad \cosh E_c = \boldsymbol{b}(\boldsymbol{p}), \qquad (65)$$

since $\rho(E, \boldsymbol{p}) < 0$ is equivalent to $a(\boldsymbol{p})/2b(\boldsymbol{p}) \leq \cosh E < b(\boldsymbol{p})$ when $\boldsymbol{p} \notin \mathcal{S}$. We will numerically estimate E_1 and E_c, the lower and upper bound of the energy interval on which the spectral density become negative. For instance, if $d = 4$, these energy values are computed as shown in the following table:

$\boldsymbol{p}$	$b(\boldsymbol{p})$	$a(\boldsymbol{p})$	$a(\boldsymbol{p})/2b(\boldsymbol{p})$	E_1	E_c
(π, π, π)	6	37	$37/12$	$1.79\ldots$	$2.48\ldots$
$(\pi, \pi, 0)$	4	17	$17/8$	$1.39\ldots$	$2.06\ldots$
$(\pi, 0, 0)$	2	5	$5/4$	$0.69\ldots$	$1.32\ldots$

Whether these values are large enough or not should depend on the physics one wants to see through the overlap boson.

IV.   REFLECTION POSITIVITY OF LATTICE WESS-ZUMINO MODEL

To remedy the violation of the reflection positivity, one may adopt the simpler nearest-neighbor action for the boson fields, ϕ and F as follows[37]:

$$S'_0 = \sum_x \left\{ -\frac{1}{2}\chi^T CD\chi + \phi^*(-\partial_\mu^*\partial_\mu)\phi + F^*F \right\}. \qquad (66)$$

This action still possesses three types of U(1) symmetry, Eq. (9), (10) and

$$\delta_\alpha\phi = +i\alpha\phi,$$
$$\delta_\alpha F = +i\alpha F, \qquad (67)$$

instead of Eq. (11).

In this formulation of the chiral multiplet, the action of the lattice $\mathcal{N} = 1$ Wess-Zumino model may be given as follows:

$$S = \sum_x \left\{ -\frac{1}{2}\chi^T CD\chi + \phi^*(-\partial_\mu^*\partial_\mu)\phi + F^*F + X^T CX \right.$$
$$\left. - g\tilde{\chi}^T C\phi P_+\tilde{\chi} - g^*\tilde{\chi}^T C\phi^* P_-\tilde{\chi} + gF\phi^2 + g^*F^*\phi^{*2} \right\}, \qquad (68)$$

where $X(x)$ is an auxiliary Majorana fermion field and $\tilde{\chi}(x) = \chi(x) + X(x)$. Then one may define the U(1)$_R$ symmetry as follows:

$$\delta_\alpha\chi = +i\alpha\gamma_5(1 - \frac{1}{2}D)\chi,$$
$$\delta_\alpha\phi = -2i\alpha\phi,$$
$$\delta_\alpha F = +4i\alpha F \qquad (69)$$

The reflection positivity is now satisfied in this formulation of the Wess-Zumino model. The θ-reflection is defined for

the bosonic fields χ, F in the same way as in the generic case (15),

$$\theta\phi(x) = \phi(\theta x) \tag{70}$$

$$\theta F(x) = F(\theta x) \tag{71}$$

and for the fermionic fields χ, X as in (16),

$$(\theta\bar{\chi})(x) = \gamma_0\chi(\theta x), \qquad (\theta\chi)(x) = \bar{\chi}(\theta x)\gamma_0, \tag{72}$$

$$(\theta\bar{X})(x) = \gamma_0 X(\theta x), \qquad (\theta X)(x) = \bar{X}(\theta x)\gamma_0. \tag{73}$$

Note that this definition of θ reflection does not contradict to the Majorana conditions $\bar{\chi} = \chi^T C$ and $\bar{X} = X^T C$. Our field algebra $\mathcal{A}$ here is that of the polynomial algebra of fermionic fields whose coefficients are the well-behaved functions of the bosonic fields. We extend θ operation to whole algebra $\mathcal{A}$, by the relations (18) and (19).

To prove the reflection positivity of the Wess-Zumino model, it is sufficient to show that the action (68) can be rewritten in the form of (30)

$$-S = B + \theta(B) + \sum_s \theta(C_s)C_s, \quad B, C_s \in \mathcal{A}_+, \tag{74}$$

[23]. Let us first consider the free part of (68). The first term in (68), the overlap Majorana fermion, can be written in the form of (30) as is shown in ref [23]. On the other hand, the second term can be written in the form of (74), as is well-known. Furthermore, the third and fourth terms in (68) is θ reflection of themselves and do not contain any 'time hopping terms'. Therefore, these terms can be written in the form of

$$\sum_{x \in \Lambda} \left\{ F^*F + X^T C X \right\}$$

$$= \sum_{x \in \Lambda_+} \left\{ F^*F + X^T C X \right\} + \theta \Big[\sum_{x \in \Lambda_+} \left\{ F^*F + X^T C X \right\} \Big]. \tag{75}$$

The rest of the terms in (68) are interaction terms,

$$S_{\text{int}} := \sum_x \Big\{ -g\tilde{\chi}^T C\phi P_+\tilde{\chi} - g^*\tilde{\chi}^T C\phi^* P_-\tilde{\chi} + $$
$$+ gF\phi^2 + g^*F^*\phi^{*2} \Big\}, \tag{76}$$

which are all strictly local. They are equal to theta-reflection of themselves again, and do not contain any nonlocal 'time hopping' terms either. This means that S_{int} can also be written as

$$S_{\text{int}} = B + \theta(B) \tag{77}$$

with

$$B = \sum_{x \in \Lambda_+} \Big\{ -g\tilde{\chi}^T C\phi P_+\tilde{\chi} - g^*\tilde{\chi}^T C\phi^* P_-\tilde{\chi} + $$
$$+ gF\phi^2 + g^*F^*\phi^{*2} \Big\}, \tag{78}$$

which obviously belongs to $\mathcal{A}_+$. Therefore, one concludes that this lattice Wess-Zumino model satisfies the reflection positivity condition.

V. DISCUSSION

Preserving R symmetry exactly is a useful way in formulating supersymmetric field theories on the lattice. This point has been emphasized by Elliot, Giedt and Moore [26] in their formulation of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. The discrete R symmetry in the two-dimensional $\mathcal{N} = 2$ Wess-Zumino model [27] has played an important role in the numerical study of the correspondence to $\mathcal{N} = 2$ conformal field theories[28].

In formulating the exact R symmetry on the lattice, however, there is a freedom in the choice of the bosonic part of the action. When one can preserve some part of the extended supersymmetries in the theories with $\mathcal{N} \geq 2$ [27, 29], it seems useful to adopt the bosonic actions to preserve the supersymmetries, although one should take into care a possible effect of the violation of the reflection positivity. But, for the theories of $\mathcal{N} = 1$, it seems difficult to preserve the supersymmetry in general[30], and the free limit supersymmetry does not necessarily help in taking the supersymmetric limit in the interacting models. In such situations, thought, if one can preserve the fundamental requirement of the reflection positivity condition, it may serve as a possible guideline to choose a bosonic action.

It would be interesting to examine further the inter-relation among the reflection positivity, the vacuum energy cancellation(the exact supersymmetry) and the exact $U(1)_R$ symmetry of free chiral multiplet on the lattice. If one adopts the Majorana Wilson fermion for the fermionic component of the chiral multiplet, one can show that the bosonic part of the supersymmetric action now fulfills the reflection positivity condition. In this case, the $U(1)_R$ symmetry is not manifest. But, through the block spin transformation, it is recovered in the fixed point action [8]. In this course of the renormalization group transformatons, it seems possible to maintain the vacuum energy cancellation by adjusting the parameters in the block-spin kernels and the normalization factors. Then, if the reflection positivity could also be maintained through the block-spin transformation, all the three conditions could be fulfills in the fixed point approach[31–35].

Appendix A: Spectral density ρ

In this appendix, we derive the formula (50) for $d = 4$. To find the explicit form of the spectral density ρ, we express the

propagator by its Fourier transformation :

$$\Delta_+(x,y) = \int \frac{d^4p}{(2\pi)^4} \mathrm{e}^{ip(x-y)} \frac{1}{\square}(p_0, \boldsymbol{p})$$
$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} \int \frac{dp_0}{2\pi} \mathrm{e}^{ip_0(x_0-y_0)} \frac{1}{\square}(p_0, \boldsymbol{p}).$$
(A1)

In the following analysis, we will apply Cauchy's theorem to the p_0 integration

$$\int \frac{dp_0}{2\pi} \mathrm{e}^{ip_0(x_0-y_0)} \frac{1}{\square}(p_0, \boldsymbol{p}).$$
(A2)

Note that though the integration (A1) is well defined, the integrand function of $\boldsymbol{p}$ in (A1):

$$\boldsymbol{p} \mapsto \int \frac{dp_0}{2\pi} \mathrm{e}^{ip_0(x_0-y_0)} \frac{1}{\square}(p_0, \boldsymbol{p})$$
(A3)

is not defined at $\boldsymbol{p} = 0$, because at $\boldsymbol{p} = 0$ this p_0 integration does not exist. Since the value of the integrand on a set of zero measure $\{(p_0, \boldsymbol{p}) : \boldsymbol{p} = 0\}$ does not contribute the integration, we may assume $\boldsymbol{p} \neq 0 \pmod{2\pi}$ in (A3) when applying Cauchy's theorem.

Define a function $f(z)$ as

$$f(z) = \frac{1}{\square}(p_0 = z, \boldsymbol{p}).$$
(A4)

From eq.(38), one finds

$$f(z) = \frac{\sqrt{a - 2b\cos z}}{\sqrt{a - 2b\cos z} + b - \cos z},$$
(A5)

where, for notational simplicity, we have written $a = a(\boldsymbol{p})$ and $b = b(\boldsymbol{p})$. Here, we have to clarify the meaning of the square root of complex variables. We define the square root of

$$z = |z|\mathrm{e}^{i\theta}, \quad \theta \in (-\pi, \pi),$$
(A6)

as

$$\sqrt{z} := \sqrt{|z|}\mathrm{e}^{i\theta/2}, \quad \theta \in (-\pi, \pi).$$
(A7)

Namely, we choose the branch where $\mathrm{Re}\,\sqrt{z} \geq 0$.

We investigate the analytic structure of f.

First, since the square root of negative real number cannot be defined, f cannot be defined where

$$a - 2b\cos z < 0.$$
(A8)

To find the more explicit condition which is equivalent to (A8), put $z = x + iy\,(x, y \in \mathbb{R})$. Since $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$, (A8) is equivalent to

$$\cos x \cosh y - i \sin x \sinh y > \frac{a}{2b},$$
(A9)

which holds true when and only when

$$\cos x \cosh y > \frac{a}{2b}, \quad \sin x \sinh y = 0.$$
(A10)

The second condition is equivalent to

$$x = n\pi\,(n \in \mathbb{Z}) \quad \text{or} \quad y = 0,$$
(A11)

but the second choice $y = 0$ is impossible because in this case the first condition of (A10) becomes

$$\cos x > \frac{a}{2b} \geq 1,$$
(A12)

which is never true for real x. Therefore, (A10) is equivalent to $x = n\pi$, $n \in \mathbb{Z}$, and

$$\cos(n\pi)\cosh y = (-1)^n \cosh y > \frac{a}{2b}, \quad (n \in \mathbb{Z}). \quad \text{(A13)}$$

Hence the condition (A8) occurs when and only when

$$x = 2n\pi\,(n \in \mathbb{Z}),$$
(A14)

and

$$y < -E_1 \quad \text{or} \quad E_1 < y.$$
(A15)

Next, we investigate the pole type singularity of f which may appear where the denominator

$$g(z) := \sqrt{a - 2b\cos z} + b - \cos z$$
(A16)

vanishes. To find the *necessary* condition of $g(z) = 0$, let us assume $g(z) = 0$. Then, by taking the square of the both sides of

$$\sqrt{a - 2b\cos z} = -b + \cos z,$$
(A17)

one finds

$$\cos^2 z = a - b^2.$$
(A18)

Using the identity $\cos^2 z = (1 + \cos 2z)/2$ and putting $z = x + iy\,(x, y \in \mathbb{R})$ again, one arrives at

$$\cos 2x \cosh 2y = 2a - 2b^2 - 1$$
(A19)

and

$$\sin 2x \sinh 2y = 0.$$
(A20)

Eq. (A20) implies

$$y = 0 \quad \text{or} \quad 2x = n\pi\,(n \in \mathbb{Z}),$$
(A21)

and we consider both cases respectively.

In the first case, $y = 0$, (A19) becomes

$$\cos 2x = 2a - 2b^2 - 1 \geq 1,$$
(A22)

which is possible only when $2a - 2b^2 - 1 = 1$, equivalently,

$$\sum_{k=1}^{3} \sin^2 p_k = 0.$$
(A23)

Therefore, this case $y = 0$ occurs only when the spacial momentum satisfies

$$\boldsymbol{p} = (m_1\pi, m_2\pi, m_3\pi), \quad m_1, m_2, m_3 \in \mathbb{Z}. \quad \text{(A24)}$$

As we noted at the beginning of this appendix, we may assume at least one of the m_k's ($k = 1, 2, 3$) is odd (otherwise the p_0 integration is ill defined). If (A24) is satisfied, the right hand side of (A22) becomes 1 and (A22) implies

$$x = n\pi, \quad n \in \mathbb{Z}. \quad \text{(A25)}$$

But, this condition, $y = 0$ and $x = n\pi$ $(n \in \mathbb{Z})$, is not sufficient for $g(z) = 0$. In fact, for $n \in \mathbb{Z}$,

$$\begin{aligned}
g(n\pi) &= \sqrt{a - 2(-1)^n b} + b - (-1)^n \\
&= |b - (-1)^n| + b - (-1)^n \\
&\geq 2
\end{aligned} \quad \text{(A26)}$$

because

$$b = b(\boldsymbol{p}) = \sum_{k=1}^{3}(1 - \cos p_k) \geq 2, \quad \text{(A27)}$$

due to the fact that at least one of $\cos p_k$'s is equal to -1.

In the second case of (A21), $2x = n\pi$ $(n \in \mathbb{Z})$, (A19) becomes

$$(-1)^n \cosh 2y = 2a - 2b^2 - 1. \quad \text{(A28)}$$

Then, this implies that n is even and

$$\cosh 2y = 2a - 2b^2 - 1. \quad \text{(A29)}$$

Define $E_0 > 0$ as

$$\begin{aligned}
E_0 &= \frac{1}{2}\cosh^{-1}(2a - 2b^2 - 1) \\
&= \cosh^{-1}\sqrt{a - b^2},
\end{aligned} \quad \text{(A30)}$$

and we obtain as a necessary condition for $g(z) = 0$, $z = n\pi \pm iE_0$ $(n \in \mathbb{Z})$. To find a sufficient condition for $g(z) = 0$, let us assume, Conversely, when $z = n\pi \pm iE_0$, then,

$$\begin{aligned}
g(n\pi \pm iE_0) &= \sqrt{a - 2b(-1)^n \cosh E_0} + b - \cosh E_0 \\
&= |b - (-1)^n\sqrt{a - b^2}| + b - \sqrt{a - b^2} \\
&= \begin{cases} 0 & (\text{if } b - (-1)^n\sqrt{a - b^2} \leq 0) \\ 2b & (\text{if } b - (-1)^n\sqrt{a - b^2} \geq 0) \end{cases}.
\end{aligned} \quad \text{(A31)}$$

Hence, the necessary and sufficient condition for $g(n\pi \pm iE_0) = 0$ is n is even and

$$b - \sqrt{a - b^2} \leq 0, \quad \text{(A32)}$$

which is equivalent to

$$1 + \sum_{k=1}^{3}\sin^2 p_k - b(\boldsymbol{p})^2 \geq 0, \quad \text{(A33)}$$

namely, $\boldsymbol{p} \in \mathcal{S}$.

We now have found all the zeros of the function $g(z)$:

$$z = 2n\pi \pm iE_0, \quad (n \in \mathbb{Z}, \boldsymbol{p} \in \mathcal{S}). \quad \text{(A34)}$$

For a moment, let us assume spacial momentum $\boldsymbol{p}$ satisfies

$$b - \sqrt{a - b^2} < 0. \quad \text{(A35)}$$

In this case, $z = z_n^\pm := 2n\pi \pm iE_0$ is a simple pole of f, as will be seen. From the above argument, f is analytic on

$$\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}}\left(\{z_n^\pm\} \cup \{2n\pi + iy\,;\, y < -E_1, E_1 < y\}\right). \quad \text{(A36)}$$

Expand g in Taylor series around $z_n^\pm$:

$$g(z) = g(z_n^\pm) + g'(z_n^\pm)(z - z_n^\pm) + \mathcal{O}\big((z - z_n^\pm)^2\big), \quad \text{(A37)}$$

on $|z - z_n^\pm| < r$ for sufficiently small $r > 0$, and we obtain

$$\begin{aligned}
f(z) &= \frac{\sqrt{a - 2b\cos z}}{g(z_n^\pm) + g'(z_n^\pm)(z - z_n^\pm) + \mathcal{O}\big((z - z_n^\pm)^2\big)} \\
&= \frac{\sqrt{a - 2b\cos z}}{g'(z_n^\pm)(z - z_n^\pm) + \mathcal{O}\big((z - z_n^\pm)^2\big)},
\end{aligned} \quad \text{(A38)}$$

on $|z - z_n^\pm| < r$. Then, we find

$$z \mapsto (z - z_n^\pm)f(z) = \frac{\sqrt{a - 2b\cos z}}{g'(z_n^\pm) + \mathcal{O}\big(z - z_n^\pm\big)} \quad \text{(A39)}$$

is analytic on $|z - z_n^\pm| < r$, and then $z = z_n^\pm$ are simple poles of f with residues

$$\begin{aligned}
\text{Res}(f, z_n^\pm) &= (z - z_n^\pm)f(z)\Big|_{z = z_n^\pm} \\
&= \frac{\sqrt{a - 2b\cos(2n\pi \pm iE_0)}}{g'(2n\pi \pm iE_0)} \\
&= \pm\frac{a - 2b\sqrt{a - b^2}}{i\sqrt{a - b^2}\sqrt{a - b^2 - 1}}.
\end{aligned} \quad \text{(A40)}$$

Applying Cauchy's theorem on the contour drawn in FIG.2., we obtain, for $x_0 > 0$,

$$\begin{aligned}
&\int_{[-\pi,\pi]}\frac{dp_0}{2\pi}\,e^{ip_0 x_0}\frac{1}{\square}(p_0, \boldsymbol{p}) \\
&= \int_{[-\pi,\pi]}\frac{dz}{2\pi}\,e^{izx_0}f(z) \\
&= 2\pi i\,\text{Res}(f, z_0) - \left(\int_{i\infty+0}^{iE_1+0} + \int_{i\infty-0}^{iE_1-0}\right)\frac{dz}{2\pi}\,e^{izx_0}f(z).
\end{aligned} \quad \text{(A41)}$$

Recalling our definition of the square root, one finds

$$\sqrt{a - 2b\cos(iE \pm 0)} = \pm i\sqrt{2b\cosh E - a}. \quad \text{(A42)}$$

Then, the integrations of the second term in (A41) is computed by putting $z = iE \pm 0$ to become

$$\left(\int_{i\infty+0}^{iE_1+0} + \int_{i\infty-0}^{iE_1-0} \right) e^{-Ex_0} f(z)$$

$$= \int_{E_1}^{\infty} \frac{idE}{2\pi} e^{-Ex_0} \Big(f(iE+0) + f(iE-0) \Big)$$

$$= - \int_{E_1}^{\infty} \frac{dE}{\pi} e^{-Ex_0} \frac{(b - \cosh E)\sqrt{2b\cosh E - a}}{\cosh^2 E - a + b^2}. \quad \text{(A43)}$$

By substituting (A40) and (A43) into (A41), we arrive at

$$\int_{[-\pi,\pi]} \frac{dp_0}{2\pi} e^{ip_0 x_0} \frac{1}{\square}(p_0, \boldsymbol{p})$$

$$= 2\pi \frac{a - 2b\sqrt{a - b^2}}{\sqrt{a - b^2}\sqrt{a - b^2 - 1}}$$

$$+ \int_{E_1}^{\infty} \frac{dE}{\pi} e^{-Ex_0} \frac{(b - \cosh E)\sqrt{2b\cosh E - a}}{\cosh^2 E - a + b^2},$$

$$\text{(A44)}$$

in agreement with (50).

Considering the case where spacial momentum $\boldsymbol{p}$ satisfies

$$b - \sqrt{a - b^2} \geq 0, \quad \text{(A45)}$$

we find that there is no pole term and only the second term of (A44) survives. Note that, in the case of equality, even though

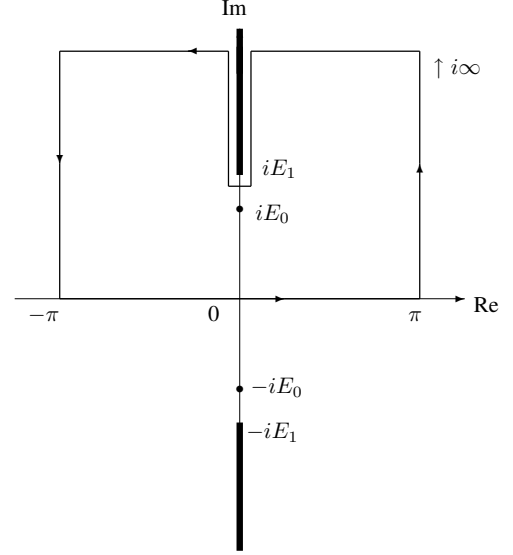$g(z) = 0$, f has no isolated pole. In this case, the numerator of $f(z)$ also vanishes and $E_0 = E_1$.

FIG. 2: Integration contour of f.

[1] J. Wess and B. Zumino, Phys. Lett. B **49**, 52 (1974).
[2] H. Neuberger, Phys. Lett. B **417**, 141 (1998) .
[3] H. Neuberger, Phys. Lett. B **427**, 353 (1998) .
[4] Y. Kikukawa and H. Neuberger, Nucl. Phys. B **513**, 735 (1998) [arXiv:hep-lat/9707016].
[5] L. H. Karsten and J. Smit, Nucl. Phys. B **183**, 103 (1981).
[6] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B **185**, 20 (1981) [Erratum-ibid. B **195**, 541 (1982)].
[7] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B **193**, 173 (1981).
[8] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D **25**, 2649 (1982).
[9] M. Luscher, Phys. Lett. B **428**, 342 (1998) .
[10] T. Aoyama and Y. Kikukawa, Phys. Rev. D **59**, 054507 (1999) [arXiv:hep-lat/9803016].
[11] K. Fujikawa and M. Ishibashi, Nucl. Phys. B **622**, 115 (2002) [arXiv:hep-th/0109156].
[12] K. Fujikawa and M. Ishibashi, Phys. Lett. B **528**, 295 (2002) [arXiv:hep-lat/0112050].
[13] K. Fujikawa, Nucl. Phys. B **636**, 80 (2002) [arXiv:hep-th/0205095].
[14] M. Bonini and A. Feo, JHEP **0409**, 011 (2004) [arXiv:hep-lat/0402034].
[15] Y. Kikukawa and H. Suzuki, JHEP **0502**, 012 (2005) [arXiv:hep-lat/0412042].
[16] M. Bonini and A. Feo, Phys. Rev. D **71**, 114512 (2005) [arXiv:hep-lat/0504010].
[17] C. Chen, E. Dzienkowski and J. Giedt, arXiv:1005.3276 [hep-lat].
[18] K. Osterwalder and R. Schrader, Commun. Math. Phys. **31**, 83 (1973).
[19] K. Osterwalder and R. Schrader, Commun. Math. Phys. **42**, 281 (1975).
[20] K. Osterwalder and E. Seiler, Annals Phys. **110**, 440 (1978).
[21] M. Luscher, Commun. Math. Phys. **54**, 283 (1977).
[22] P. Menotti and A. Pelissetto, Nucl. Phys. Proc. Suppl. **4**, 644 (1988).
[23] Y. Kikukawa and K. Usui, Phys. Rev. D **82**, 114503 (2010) [arXiv:1005.3751 [hep-lat]].
[24] M. Luscher, arXiv:hep-th/0102028.
[25] K. Usui, In preparation.
[26] J. W. Elliott, J. Giedt and G. D. Moore, Phys. Rev. D **78**, 081701 (2008) [arXiv:0806.0013 [hep-lat]].
[27] Y. Kikukawa and Y. Nakayama, Phys. Rev. D **66**, 094508 (2002).
[28] H. Kawai and Y. Kikukawa, Phys. Rev. D **83**, 074502 (2011) [arXiv:1005.4671 [hep-lat]].
[29] Y. Kikukawa and F. Sugino, Nucl. Phys. B **819**, 76 (2009) [arXiv:0811.0916 [hep-lat]].
[30] M. Kato, M. Sakamoto and H. So, JHEP **0805**, 057 (2008).
[31] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B **427**, 125 (1998) [arXiv:hep-lat/9801021].
[32] P. Hasenfratz, Nucl. Phys. B **525**, 401 (1998) [arXiv:hep-lat/9802007].
[33] T. L. Bell and K. G. Wilson, Phys. Rev. **B11** (1975) 3431
[34] U. J. Wiese, Phys. Lett. B **315**, 417 (1993) [arXiv:hep-

lat/9306003].

[35] H. So and N. Ukita, Phys. Lett. B **457**, 314 (1999) [arXiv:hep-lat/9812002].

[36] In the following, we write the bosonic field argument of a function like $S(\phi)$ instead of $S(\phi, \phi^*)$ for the notational simplicity. This notation never means that S is an analytic function of ϕ.

[37] Here, we have changed the sign convention of the fermionic action by introducing new Majorana field $\chi' = i\chi$. Of course this does not change any physical results. It is simply because this convention has been used in the proof of the reflection positivity for the overlap fermions in our previous work [23].